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1983 J. Phys. A: Math. Gen. 16 L133

(http://iopscience.iop.org/0305-4470/16/5/001)

J. Phys. A: Math. Gen. 16 (1983) L133–L135. Printed in Great Britain

LETTER TO THE EDITOR

Contact symmetries of the harmonic oscillator

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Received 4 January 1983

Abstract. The symmetry group of contact transformations for the one-dimensional harmonic oscillator is determined. Its generators contain two arbitrary functions, i.e. it is an infinite-parameter Lie group.

The complete symmetry group of point transformations for the one-dimensional harmonic oscillator has recently been discussed by several authors (Anderson and Davison 1974, Wulfman and Wybourne 1976, Lutzky 1978). It has been shown to be the eight-parameter group SL(3, R). It is the purpose of this paper to point out that there is a much larger group of contact transformations leaving this equation invariant. The group of point transformations referred to above is recovered as a subgroup of this latter group. The theory of contact transformations which is needed here may be found in Lie (1894). Let a one-parameter group of contact transformations be given by

$$\tilde{x} = f(x, y, p, \varepsilon) \qquad \tilde{y} = g(x, y, p, \varepsilon) \qquad \tilde{p} = h(x, y, p, \varepsilon) \tag{1}$$

with the initial conditions $x = f(x, y, p, \varepsilon = 0)$, $y = g(x, y, p, \varepsilon = 0)$ and $p = h(x, y, p, \varepsilon = 0)$; as usual $p = y'$. For the functions f, g and h to define a contact transformation it is necessary that they obey certain involution relations which are given by Lie (1894, ch 4). For our purpose it is more appropriate to work with infinitesimal contact transformations. To this end the infinitesimals ξ, η and ζ are defined by

$$\begin{aligned} &\tilde{x} = x + \varepsilon\xi(x, y, p) + \mathrm{O}(\varepsilon^2) \qquad \tilde{y} = y + \varepsilon\eta(x, y, p) + \mathrm{O}(\varepsilon^2) \\ &\tilde{p} = p + \varepsilon\zeta(x, y, p) + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{2}$$

They determine an infinitesimal contact transformation if it is possible to write them in the form

$$\xi = -W_p \qquad \eta = W - pW_p \qquad \zeta = W_x + pW_y \tag{3}$$

where the characteristic function $W(x, y, p)$ is a completely arbitrary function of its arguments. If W is linear in p the corresponding contact transformation is an extended point transformation and it holds that

$$W(x, y, p) = \eta(x, y) - p\xi(x, y). \tag{4}$$

A second-order differential equation

$$\omega(x, y, y' = p, y'') = 0 \tag{5}$$

is said to be invariant under a contact transformation if

$$\xi\, \partial\omega/\partial x + \eta\, \partial\omega/\partial y + \zeta\, \partial\omega/\partial p + \zeta_2\, \partial\omega/\partial y'' = 0 \tag{6}$$

on the manifold $\omega = 0$ in the space of the variables x, y, p and y''. The function ζ_2 is defined by

$$\zeta_2 = \frac{\partial\zeta}{\partial x} + p\frac{\partial\zeta}{\partial y} + y''\frac{\partial\zeta}{\partial p} - y''\left(\frac{\partial\xi}{\partial x} + p\frac{\partial\xi}{\partial y} + y''\frac{\partial\xi}{\partial p}\right). \tag{7}$$

In appropriate units the equation of the linear harmonic oscillator may be written as

$$y'' + y = 0. \tag{8}$$

Equations (3), (6) and (8) lead to the following linear partial differential equation for the characteristic function W:

$$W_{xx} + 2pW_{xy} + p^2 W_{yy} - 2yW_{px} - 2ypW_{py} + y^2 W_{pp} - yW_y - pW_p + W = 0. \tag{9}$$

It may be written as

$$(\partial_x + p\,\partial_y - y\,\partial_p)^2 W + (1 - y\,\partial_y - p\,\partial_p)W = 0. \tag{10}$$

Applying the well known methods for solving linear partial differential equations to (9), (10) (see e.g. Tychonoff and Samarski 1959), the general solution is obtained in the form

$$W(x, y, p) = pA(u, v) + yB(u, v) \tag{11}$$

where

$$u = (y^2 + p^2)^{1/2} \qquad v = p \sin x - y \cos x \tag{12}$$

and A, B are two arbitrary functions. So (8) allows an infinite-parameter Lie group

Table 1. The functions A and B which determine the generators $X_1 \ldots X_8$ of Wulfman and Wybourne are given.

X_i	$\xi(x, y)$	$\eta(x, y)$	$A(u, v)$	$B(u, v)$
X_1	$-y \cos x$	$(1+y^2)\sin x$	$-\frac{v}{u^2}$	$\frac{1+u^2}{u} \sin\cos^{-1}\frac{v}{u}$
X_2	$y \cos x$	$(1-y^2)\sin x$	$\frac{v}{u^2}$	$\frac{1-u^2}{u} \sin\cos^{-1}\frac{v}{u}$
X_3	$y \sin x$	$(1+y^2)\cos x$	$\frac{1}{u}\sin\cos^{-1}\frac{v}{u}$	$-v\left(1+\frac{1}{u^2}\right)$
X_4	$-y \sin x$	$(1-y^2)\cos x$	$\frac{1}{u}\sin\cos^{-1}\frac{v}{u}$	$-v\left(1-\frac{1}{u^2}\right)$
X_5	1	0	-1	0
X_6	0	y	0	1
X_7	$\sin 2x$	$y \cos 2x$	$-\sin 2\cos^{-1}\frac{v}{u}$	$\cos 2\cos^{-1}\frac{v}{u}$
X_8	$\cos 2x$	$-y \sin 2x$	$\cos 2\cos^{-1}\frac{v}{u}$	$\sin 2\cos^{-1}\frac{v}{u}$

of contact transformations leaving it invariant. Its infinitesimal generators are defined by (3) and (11). By appropriate choice of the functions A and B the eight infinitesimal generators of point symmetries as determined by Wulfman and Wybourne are obtained. They are given in table 1.

The physical meaning of this larger group of contact transformations does not seem to be obvious. It is interesting to note that the arguments of the two functions A, B are two independent first integrals of (8). It may be instructive to determine the symmetry group of contact transformations for more general equations, e.g. the anharmonic oscillator or the harmonic oscillator with damping.

References

Anderson R L and Davison S M 1974 *J. Math. Anal. Applic.* **48** 301

Lie S 1894 *Geometrie der Beruehrungstransformationen* (Leipzig: Teubner) (1977, New York: Chelsea)

Lutzky M J 1978 *J. Phys. A: Math. Gen.* **11** 249

Tychonoff A N and Samarski A A 1959 *Differentialgleichungen der mathematischen Physik* (Berlin: VEB Deutscher Verlag der Wissenschaften)

Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* **9** 507